

Math 201 — Fall 2009–10
Calculus and Analytic Geometry III, sections 1–8, 24–26
Quiz 2, December 2 — Duration: 1 hour

GRADES:

1 (/15)	2 (/15)	3 (/15)	4 (/16)	5 (/19)	6 (/20)	TOTAL/100

YOUR NAME:

Solutions

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1	Section 2	Section 3	Section 4
Lecture MWF 3	Lecture MWF 3	Lecture MWF 3	Lecture MWF 3
Professor Makdisi	Professor Makdisi	Professor Makdisi	Professor Makdisi
Recitation F 11	Recitation F 2	Recitation F 4	Recitation F 9
Ms. Nassif	Ms. Nassif	Ms. Nassif	Ms. Nassif
Section 5	Section 6	Section 7	Section 8
Lecture MWF 10	Lecture MWF 10	Lecture MWF 10	Lecture MWF 10
Professor Raji	Professor Raji	Professor Raji	Professor Raji
Recitation T 11	Recitation T 3:30	Recitation T 8	Recitation T 2
Professor Raji	Ms. Itani	Ms. Itani	Ms. Itani
Section 24	Section 25	Section 26	
Lecture MWF 2	Lecture MWF 2	Lecture MWF 2	
Professor Tlas	Professor Tlas	Professor Tlas	
Recitation F 11	Recitation F 12	Recitation F 3	
Dr. Yamani	Dr. Yamani	Professor Tlas	

INSTRUCTIONS:

1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside the booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit.
3. You may use the back of each page for scratchwork OR for solutions. There are three extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. Closed book and notes. NO CALCULATORS ALLOWED. Turn OFF and put away any cell phones.

GOOD LUCK!

An overview of the exam problems.

Take a minute to look at all the questions, THEN
solve each problem on its corresponding page INSIDE the booklet.

1. Let the function $f(x)$ be given by

$$f(x) = \begin{cases} 0, & \text{when } 0 \leq x < \pi \\ x - \pi, & \text{when } \pi \leq x < 2\pi \\ \text{and } f(x) \text{ is periodic with period } 2\pi. & \end{cases}$$

- a) (5 pts) Sketch the graph of $f(x)$ for $x \in [-2\pi, 4\pi]$.
 b) (10 pts) The Fourier series of $f(x)$ is $\sum_{n \geq 0} a_n \cos nx + \sum_{n \geq 1} b_n \sin nx$. Find ONLY the coefficients b_n .
2. a) (6 pts) Plot the polar graph of the curve $C : r = 1 + \sin \theta$. Also draw the line $L : y = 4/9$ on your graph.
 b) (3 pts) Convert the equation of L to polar coordinates.
 c) (6 pts) Find the (r, θ) -coordinates of the two points of intersection on $L \cap C$.
3. Consider the following moving point in space:

$$P(t) = (3t, \sqrt{6} e^t, \frac{1}{2} e^{2t}).$$

- a) (5 pts) Find the velocity and the speed of $P(t)$ at the instant $t = 0$.
 b) (5 pts) What is the arclength of the curve given by $P(t)$ for $0 \leq t \leq \ln 5$? Simplify your answer.
 c) (5 pts) Suppose we have a function $f(x, y, z)$ with the property

$$\vec{\nabla} f|_{(3, \sqrt{6} e, \frac{1}{2} e^2)} = (e^2, -\sqrt{6} e, 5).$$

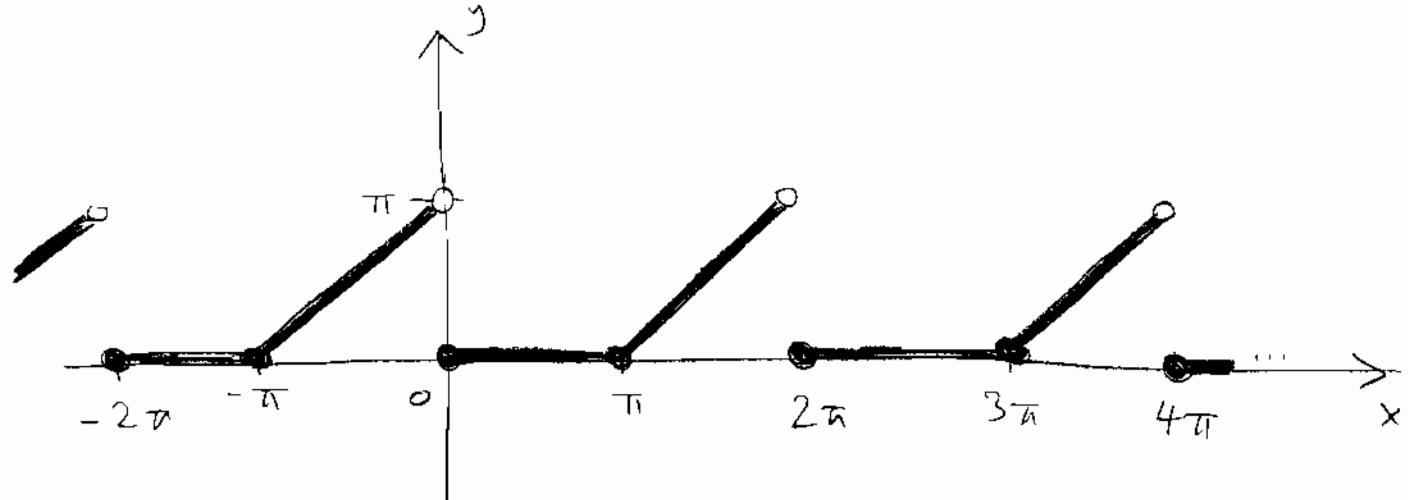
Find $\frac{d}{dt} f(P(t))$ at the instant when the point $P(t)$ passes through $(3, \sqrt{6} e, \frac{1}{2} e^2)$.

4. a) (8 pts) Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$ does not exist.
 b) (8 pts) Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2}$ does exist (hint: the limit is eqnal to 0).
5. Consider the function $f(x, y, z) = z e^{x^3 y}$.
 a) (6 pts) Find the gradient of $f(x, y, z)$ at $P_0(1, 1, -1)$.
 b) (7 pts) Find the equation of the tangent plane to the surface $f(x, y, z) = -e$ at P_0 .
 c) (6 pts) Determine the direction in which $f(x, y, z)$ increases most rapidly when the point (x, y, z) moves away from P_0 . Your answer should be a **unit vector**.
6. Given a function $f(x, y)$ satisfying $f(1, 2) = 4$, $\vec{\nabla} f|_{(1,2)} = (3, 4)$.
 a) (6 pts) Approximately how much is $f(1.03, 1.99)$?
 b) (7 pts) Find a direction \vec{u} in which the directional derivative $D_{\vec{u}} f|_{(1,2)} = 0$. Your answer \vec{u} should be a **unit vector**.
 c) (7 pts) Let S be the graph of f . In other words, $S = \{(x, y, z) \in \mathbf{R}^3 \mid z = f(x, y)\}$. Find the equation of the tangent plane to S at the point $P_0(1, 2, 4) \in S$. (Be careful.)

1. Let the function $f(x)$ be given by

$$f(x) = \begin{cases} 0, & \text{when } 0 \leq x < \pi \\ x - \pi, & \text{when } \pi \leq x < 2\pi \\ \text{and } f(x) \text{ is periodic with period } 2\pi. & \end{cases}$$

a) (5 pts) Sketch the graph of $f(x)$ for $x \in [-2\pi, 4\pi]$.



b) (10 pts) The Fourier series of $f(x)$ is $\sum_{n \geq 0} a_n \cos nx + \sum_{n \geq 1} b_n \sin nx$. Find ONLY the coefficients b_n .

$$b_n = \frac{1}{\pi} \int_{x=0}^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \left[\int_{x=0}^{\pi} 0 \sin nx dx + \int_{x=\pi}^{2\pi} (\pi - x) \sin nx dx \right]$$

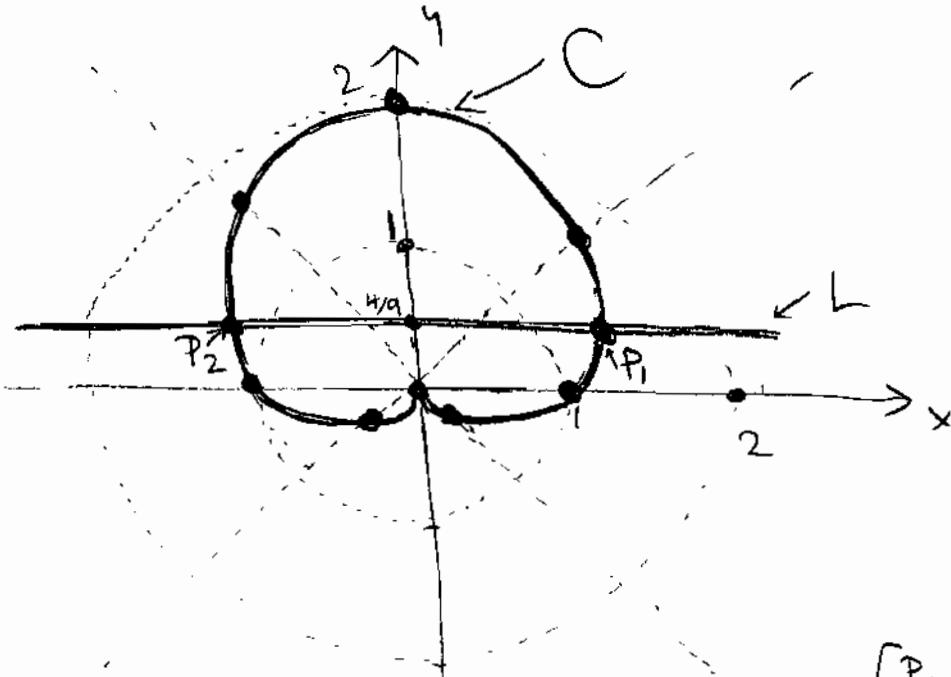
(any other interval of length 2π can be used instead of $[0, 2\pi]$)

$$\begin{aligned} &= \frac{1}{\pi} \int_{x=\pi}^{2\pi} (\pi - x) d\left(-\frac{\cos nx}{n}\right) = \frac{1}{\pi} \left\{ \left[(\pi - x) \left(-\frac{\cos nx}{n} \right) \right]_{x=\pi}^{x=2\pi} - \int_{x=\pi}^{2\pi} \left(-\frac{\cos nx}{n} \right) d(\pi - x) \right\} \\ &= \frac{1}{\pi} \left\{ \left(-\pi \left(-\frac{\cos 2\pi}{n} \right) \right) - \left(0 \left(-\frac{\cos \pi}{n} \right) \right) - \int_{x=\pi}^{2\pi} \frac{\cos nx}{n} dx \right\} \end{aligned}$$

Note $d(\pi - x) = -dx$

$$\begin{aligned} &= \frac{1}{\pi} \left(+\pi \cdot \frac{1}{n} - 0 - \left[\frac{\sin nx}{n^2} \right]_{x=\pi}^{x=2\pi} \right) = \frac{1}{\pi} \left(\frac{\pi}{n} - \frac{\sin 2\pi}{n^2} + \frac{\sin \pi}{n^2} \right) \\ &= \frac{1}{\pi} \left(\frac{\pi}{n} \right) = \boxed{\frac{1}{n}} \end{aligned}$$

2. a) (6 pts) Plot the polar graph of the curve $C : r = 1 + \sin \theta$. Also draw the line $L : y = 4/9$ on your graph.



θ	r
0	1
$\pi/4$	$1 + \sqrt{2}/2 \approx 1.7$
$\pi/2$	2
$3\pi/4$	$1 + \sqrt{2}/2 \approx 1.7$
π	1
$5\pi/4$	$1 - \sqrt{2}/2 \approx 0.3$
$3\pi/2$	0
$7\pi/4$	$1 - \sqrt{2}/2 \approx 0.3$
2π	1

[P_1 & P_2 are for part c below]

- b) (3 pts) Convert the equation of L to polar coordinates.

$$L: y = 4/9 \text{ means } r \sin \theta = 4/9$$

or
$$r = \frac{4}{9 \sin \theta}$$

- c) (6 pts) Find the (r, θ) -coordinates of the two points of intersection on $L \cap C$.

equate $r = 1 + \sin \theta = \frac{4}{9 \sin \theta}$

so $\sin \theta + \sin^2 \theta = \frac{4}{9}$, ie $\sin \theta$ is a root of $v + v^2 = \frac{4}{9}$

solve $v^2 + v - \frac{4}{9} = 0$: $v = \frac{-1 \pm \sqrt{1 + \frac{16}{9}}}{2} = \frac{-1 \pm \frac{5}{3}}{2} = -\frac{4}{3} \text{ or } \frac{1}{3}$

(or : factor $9v^2 + 9v - 4 = (3v+4)(3v-1) = 0$)

$\sin \theta = v = -\frac{4}{3}$ is impossible since $\sin \theta \in [-1, 1]$

$\sin \theta = \frac{1}{3}$ is possible & gives rise to two values of θ :

① $\theta = \sin^{-1}(\frac{1}{3})$, $r = 1 + \sin \theta = \frac{4}{3}$

② $\theta = \pi - \sin^{-1}(\frac{1}{3})$, $r = 1 + \sin \theta = \frac{4}{3}$

so
$$P_1 \left(\frac{4}{3}, \sin^{-1}\left(\frac{1}{3}\right) \right)$$

so
$$P_2 \left(\frac{4}{3}, \pi - \sin^{-1}\left(\frac{1}{3}\right) \right)$$

in polar coordinates,

3. Consider the following moving point in space:

$$P(t) = (3t, \sqrt{6}e^t, \frac{1}{2}e^{2t}).$$

a) (5 pts) Find the velocity and the speed of $P(t)$ at the instant $t = 0$.

$$\vec{r} = \overrightarrow{OP} = (3t, \sqrt{6}e^t, \frac{1}{2}e^{2t})$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (3, \sqrt{6}e^t, e^{2t})$$

at $t=0$, $\vec{v}|_{t=0} = (3, \sqrt{6}, 1)$ is the velocity

and $|\vec{v}|_{t=0} = \sqrt{9+6+1} = 4$ is the speed.

b) (5 pts) What is the arclength of the curve given by $P(t)$ for $0 \leq t \leq \ln 5$? Simplify your answer.

$$\begin{aligned} \text{The arclength is } & \int_{t=0}^{\ln 5} |\vec{v}| dt = \int_{t=0}^{\ln 5} \sqrt{9+6e^{2t}+e^{4t}} dt = \int_{t=0}^{\ln 5} \sqrt{(3+e^{2t})^2} dt \\ & = \int_{t=0}^{\ln 5} (3+e^{2t}) dt = \left[3t + \frac{e^{2t}}{2} \right]_{t=0}^{\ln 5} = 3\ln 5 + \frac{e^{2\ln 5}}{2} - 0 - \frac{1}{2} \\ & = 3\ln 5 + \frac{5^2}{2} - \frac{1}{2} = \boxed{3\ln 5 + 12}. \end{aligned}$$

c) (5 pts) Suppose we have a function $f(x, y, z)$ with the property

$$\vec{\nabla}f|_{(3, \sqrt{6}e, \frac{1}{2}e^2)} = (e^2, -\sqrt{6}e, 5).$$

Find $\frac{d}{dt}f(P(t))$ at the instant when the point $P(t)$ passes through $(3, \sqrt{6}e, \frac{1}{2}e^2)$.

$$\text{By the chain rule, } \frac{df(P(t))}{dt} = \vec{\nabla}f|_{P(t)} \cdot \vec{v}$$

here $P(t)$ passes through $P_0(3, \sqrt{6}e, \frac{1}{2}e^2)$ when $t=t_0=1$

$(3t, \sqrt{6}e^t, \frac{1}{2}e^{2t})$

from $\vec{v} = (3, \sqrt{6}e^t, e^{2t})$

so the answer is

$$\begin{aligned} \frac{df(P(t))}{dt} \Big|_{t=1} &= \vec{\nabla}f|_{(3, \sqrt{6}e, \frac{1}{2}e^2)} \cdot \vec{v}|_{t=1} = (e^2, -\sqrt{6}e, 5) \cdot (3, \sqrt{6}e, e^2) \\ &= (e^2)(3) - 6e^2 + 5e^2 = \boxed{3[2e^2]}. \end{aligned}$$

4. a) (8 pts) Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4 + y^2}$ does not exist.

Use the two-path test or parabolas of the form $y = ax^2$.

Let us express this using a path $P_a(t) = (t, at^2)$,

when $t \rightarrow 0$, $P_a(t) \rightarrow (0,0)$.

However $\lim_{t \rightarrow 0} f(P_a(t)) = \lim_{t \rightarrow 0} \frac{(t^2)(at^2)}{t^4 + (at^2)^2}$

$$= \lim_{t \rightarrow 0} \frac{at^4}{t^4(1+a^2)} = \lim_{t \rightarrow 0} \frac{a}{1+a^2} = \frac{a}{1+a^2}.$$

This limit depends on a , so approaching $(0,0)$ along two different paths $P_a(t)$ yields different values of $\lim_{t \rightarrow 0} f(P_a(t))$.

Therefore, the limit does NOT exist by the two-path test.

b) (8 pts) Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2 + y^2}$ does exist (hint: the limit is equal to 0).

We want to show that $\left| \frac{x^2y}{x^2+y^2} - 0 \right| \rightarrow 0$ when $(x,y) \rightarrow (0,0)$.

first way use polar coordinates $x = r\cos\theta$ $y = r\sin\theta$ $r = \text{distance from } (x,y) \text{ to } (0,0)$

$$0 \leq \underbrace{\left| \frac{r^3 \cos^2\theta \sin\theta}{r^2} \right|}_{\text{error}} = r |\cos^2\theta \sin\theta| \leq r \quad \text{and } r \rightarrow 0 \\ \text{so the error} \rightarrow 0. \underline{\text{DONE.}}$$

second way $P_0 = (0,0)$, $P = (0+\Delta x, 0+\Delta y) = (\Delta x, \Delta y)$, $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} \geq |\Delta x|, |\Delta y|$

$$|\text{error}| = \left| \frac{(\Delta x)^2 \Delta y}{(\Delta s)^2} \right| \leq \left| \frac{(\Delta s)^2 |\Delta s|}{(\Delta s)^2} \right| = \Delta s$$

so when $\Delta s \rightarrow 0$, the error $\rightarrow 0$.

third way $0 \leq x^2 \leq x^2 + y^2 \Leftrightarrow 0 \leq \frac{x^2}{x^2 + y^2} \leq 1$,

$$\text{so } |\Delta y| \leq y \cdot \left(\frac{x^2}{x^2 + y^2} \right) \leq |\Delta y| \quad (\text{use } |\Delta y| \text{ since } y \text{ can be +ve or -ve})$$

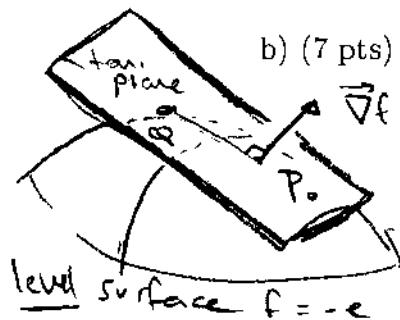
and $\lim_{(x,y) \rightarrow (0,0)} |\Delta y| = 0 = \lim_{(x,y) \rightarrow (0,0)} -|\Delta y|$. Now use the sandwich theorem.

5. Consider the function $f(x, y, z) = ze^{x^3y}$.

a) (6 pts) Find the gradient of $f(x, y, z)$ at $P_0(1, 1, -1)$.

$$\vec{\nabla}f = (f_x, f_y, f_z) = \left(z \cdot 3x^2 e^{x^3y}, z \cdot x^3 e^{x^3y}, e^{x^3y} \right).$$

$$\begin{aligned} \text{so } \vec{\nabla}f \Big|_{(1,1,-1)} &= (-1) \cdot 3 \cdot e^1, (-1) \cdot 1 \cdot e^1, e^1 \\ &= \boxed{(-3e, -e, e)}, \end{aligned}$$



b) (7 pts) Find the equation of the tangent plane to the surface $f(x, y, z) = -e$ at P_0 .

$\vec{\nabla}f|_{P_0}$ is \perp tan. plane
 $Q \in$ the tan. plane $\Leftrightarrow \overrightarrow{P_0Q} \perp \vec{\nabla}f|_{P_0}$, meaning
 $\overrightarrow{P_0Q} \cdot \vec{\nabla}f|_{P_0} = 0$

$$\Leftrightarrow (x-1, y-1, z+1) \perp (-3e, -e, e),$$

$$\text{meaning } (-3e)(x-1) - e(y-1) + e(z+1) = 0$$

\Leftrightarrow

(cancel "e" factor
& simplify)

$$\boxed{-3x - y + z = -5}$$

c) (6 pts) Determine the direction in which $f(x, y, z)$ increases most rapidly when the point (x, y, z) moves away from P_0 . Your answer should be a unit vector.

The direction of most rapid increase is that of $\vec{\nabla}f|_{P_0} = (-3e, -e, e)$.

To make this a unit vector, we take

$$\begin{aligned} \vec{v} &= \frac{\vec{\nabla}f|_{P_0}}{|\vec{\nabla}f|_{P_0}|} = \frac{(-3e, -e, e)}{\sqrt{9e^2 + e^2 + e^2}} = \frac{(-3e, -e, e)}{\sqrt{11}e} \\ &= \boxed{\left(\frac{-3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right)}. \end{aligned}$$

6. Given a function $f(x, y)$ satisfying $f(1, 2) = 4$, $\vec{\nabla}f \Big|_{(1,2)} = (3, 4)$.

a) (6 pts) Approximately how much is $f(1.03, 1.99)$?

$$P_0 = (1, 2), P = (1.03, 1.99), \Delta \vec{r} = \vec{P_0 P} = (\Delta x, \Delta y) = (0.03, -0.01).$$

$$\text{we have } f(P) = f(P_0) + \Delta f \approx f(P_0) + \vec{\nabla}f|_{P_0} \cdot \Delta \vec{r}$$

$$= 4 + (3, 4) \cdot (0.03, -0.01) \\ = 4 + 0.09 - 0.04 = \boxed{4.05} \quad \begin{matrix} \text{approximate} \\ \text{value of } f(P) \end{matrix}$$

Note this is the same as writing

$$f(P) \approx f(P_0) + f_x|_{P_0} \Delta x + f_y|_{P_0} \Delta y.$$

You can also solve this using the directional derivative, but it takes longer.

b) (7 pts) Find a direction \vec{u} in which the directional derivative $D_{\vec{u}} f \Big|_{(1,2)} = 0$. Your answer \vec{u} should be a unit vector.

here $D_{\vec{u}} f \Big|_{(1,2)} = \vec{\nabla}f \Big|_{(1,2)} \cdot \vec{u} = (3, 4) \cdot \vec{u} \stackrel{\text{we want}}{=} 0$

so we want $\vec{u} \perp (3, 4)$, \vec{u} another \vec{u} $\rightarrow (3, 4)$

note that $(3, 4) \perp (4, -3)$

(check $(3, 4) \cdot (4, -3) = 12 - 12 = 0$) $\rightarrow (4, -3)$

but $(4, -3)$ is not a unit vector.

we can normalize it however:

so $\boxed{\vec{u} = \frac{(4, -3)}{\|(4, -3)\|} = \frac{(4, -3)}{5} = \left(\frac{4}{5}, -\frac{3}{5}\right)}$ is one solution.
(the other solution is $\vec{u} = \left(-\frac{4}{5}, \frac{3}{5}\right)$.)

c) (7 pts) Let S be the graph of f . In other words, $S = \{(x, y, z) \in \mathbf{R}^3 \mid z = f(x, y)\}$.
Find the equation of the tangent plane to S at the point $P_0(1, 2, 4) \in S$. (Be careful.)

The idea here is to make the set S into a level set (not a graph!).

Introduce the function $g(x, y, z) = f(x, y) - z$.

Then S is the level set $\{(x, y, z) \mid g(x, y, z) = 0\}$.

$$\vec{\nabla}g|_{P_0} = (f_x, f_y, -1) \Big|_{(x, y, z) = (1, 2, 4)} = (3, 4, -1)$$

$\vec{\nabla}g|_{P_0} = (3, 4, -1)$

from $\vec{\nabla}f|_{(x, y) = (1, 2)} = (3, 4)$

so we want the plane $\perp (3, 4, -1)$ passing through $P_0(1, 2, 4)$.

The equation is $3x + 4y - z = 7$

[Alternate way: use the linear approx in a) : $z = 4 + 3(x-1) + 4(y-2)$]

for details,
see
question 5,
part b